JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 212–225 (1996)
ARTICLE NO. 0251

Duality for Vector Optimization of Set-Valued Functions

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Submitted by Koichi Mizukami

Received May 15, 1995

In this note, a general cone separation theorem between two subsets of image space is presented. With the aid of this, optimality conditions and duality for vector optimization of set-valued functions in locally convex spaces are discussed.

1. INTRODUCTION

Since many optimization problems encountered in economics and other fields involve set-valued constraints and set-valued objective functions, vector optimization problems for set-valued functions have received an increasing amount of attention in recent years. In particular, Lagrangian conditions and duality of convex and weakened convex set-valued functions have been discussed under some constraint qualifications, especially Slater's condition (see [5, 13–15, 18–20]). Luc and Jahn [14] studied an axiomatic approach which allows us to obtain duality theorems for nonconvex vector optimizations. Recently, a kind of regularity condition via the image space approach was developed by Giannessi [8] in studying constrained scalar optimization, and it was further investigated by other authors in studying scalar and vector optimization in finite dimensional spaces (see [7, 16]). The image space approach has been proved to be a fruitful method in many topics of optimization theory (e.g., optimality condition, existence of solution, duality, and stability) (see [5, 7–9, 12, 16, 17, 20–22]).

We observe that the optimality of a feasible point is equivalent to the separation between two suitable subsets of the image space. The separa-

0022-247X/96 $18.00

tion of two convex sets by a hyperplane is achieved under some conditions. When the sets are not convex, there is no guarantee that separation by a hyperplane is possible. Henig [10] established a separation by a cone in a finite dimensional space, which includes, as a special case, hyperplane separation. The cone separation was also studied by Dien, Mastroeni, Pappalardo, and Quang [7], Jahn [11], and Dauer and Saleh [6] in finite dimensional spaces, norm spaces, and locally convex spaces, respectively.

In this paper, we investigate cone separation between two suitable subsets of the image space, applying the results obtained to study optimality conditions and duality for weakened convex and nonconvex set-valued functions in locally convex spaces. The results obtained are based on efficient solutions (minimum solutions) in locally convex spaces.

2. PRELIMINARIES

Let X, Y, and Z be real locally convex topological vector spaces with topological dual spaces X^*, Y^*, and Z^*, respectively. Let $S \subset Y$, $Q \subset Z$ be pointed closed convex cones. The dual cone S^+ and its quasi-interior S^{+i} are defined as

$$S^+ = \{y^* \in Y^* | \langle y^*, y\rangle \geq 0, \forall y \in S\},$$

and

$$S^{+i} = \{y^* \in Y^* | \langle y^*, y\rangle > 0, \forall y \in S \setminus \{0\}\},$$

where $\langle\ ,\ \rangle$ is the canonical bilinear form with respect to the duality between Y^* and Y.

We say that a subset B of S is a *base* for S if B is convex, $0 \notin \overline{B}$, and $S = \text{cone}(B) = \{\lambda b \mid \lambda \geq 0,\ b \in B\}$.

It is easy to show that if S has a base, then S^{+i} is nonempty (see [3]).

A functional ϕ: $Y \to R$ is called *S-increasing* (resp. *S-strictly increasing*) if $y_1, y_2 \in Y$, $y_1 - y_2 \in S$ implies $\phi(y_1) \geq \phi(y_2)$ (and $y_1 - y_2 \in S \setminus \{0\}$ implies $\phi(y_1) > \phi(y_2)$ resp.).

Suppose that A is a convex set. A set-valued function F: $X \to 2^Y$ is said to be convex on A, if for any $x_1, x_2 \in A$, $\lambda \in [0,1]$

$$\lambda F(x_1) + (1 - \lambda) F(x_2) \subset F(\lambda x_1 + (1 - \lambda) x_2) + S.$$

A set-valued function F: $X \to 2^Y$ is said to be *nearly convexlike* on A, if $\overline{F(A) + S}$ is convex.

It is obvious that if F is convex on A, then F is nearly convexlike on A. But the converse is not true, i.e., a nearly convexlike set-valued function is not necessarily convex.

EXAMPLE. Let $X = Y = R^1$, $A = [-1,1]$, F: $A \to R^1$ be a set-valued function defined by

$$F(x) = \begin{cases} [0,1], & \text{if } x \in [-1,0]; \\ [-1,0], & \text{if } x \in (0,1]. \end{cases}$$

It is obvious that F is not convex on A, but $F(A) + R^1_+ = \{r \in R^1 \mid r \geq -1\}$ is convex. This means that F is nearly convexlike on A.

Let $A \subset X$ be a subset, and F: $X \to 2^Y$, G: $X \to 2^Z$ be set-valued functions.

We consider the following vector optimization problems

$$\min F(x), \tag{P}$$

$$\text{subject to } x \in A, G(x) \cap (-Q) \neq \emptyset. \tag{2.1}$$

We say that x is a *feasible point* for problem (P) if x satisfies (2.1). The set of all such points is denoted by E. A point $x_0 \in E$ is said to be a *minimum point* for problem (P) if there exists $y_0 \in F(x_0)$ such that there is no $x \in E$ satisfying $(F(x) - y_0) \cap (-S \setminus \{0\}) \neq \emptyset$; we say that y_0 is a *minimum value* for (P) and call (x_0, y_0) a *minimum solution* of problem (P). These definitions are consistent with those of Corley [5] (see also [13, 15, 18, 19]).

If $(x_0, y_0) \in \operatorname{gr} F = \{(x, y) \mid y \in F(x)\}$ satisfies

$$\overline{\operatorname{cone}}(F(E) + S - y_0) \cap (-S) = \{0\}, \tag{2.2}$$

where $\overline{\operatorname{cone}}(B - y)$ denotes the closure of the cone generated by $B - y$, we say that (x_0, y_0) is a *Benson's proper minimum solution* of (P) (see [2]).

Set $H_{y_0}(x) = (F(x) - y_0, G(x))$, $C = (S \setminus \{0\}) \times Q$, $R_{y_0} = H_{y_0}(A) + S \times Q$. $H_{y_0}(A)$ and R_{y_0} will be called the *image* and *extended image* of problem (P).

PROPOSITION 2.1. *If $x_0 \in E$ is a feasible point of problem* (P) *and $(x_0, y_0) \in \operatorname{gr} F$, then the following statements are equivalent*

(i) (x_0, y_0) *is a minimum solution of* (P);

(ii) $H_{y_0}(A) \cap (-C) = \emptyset$;

(iii) $R_{y_0} \cap (-C) = \emptyset$;

(iv) $R_{y_0} \cap [-(S \setminus \{0\}) \times \{0\}] = \emptyset$.

Proof. (i) $\Rightarrow$ (ii). Assume the contrary. Then there exist $s \in S \setminus \{0\}$, $q \in Q$, and $x \in A$ such that $-(s, q) \in (F(x) - y_0, G(x))$. This means that $x \in E$ and $(F(x) - y_0) \cap (-S \setminus \{0\}) \neq \emptyset$, a contradiction.

(ii) $\Rightarrow$ (iii). Assume the contrary. There exist $s \in S \setminus \{0\}$, $q \in Q$, and $x \in A$ such that $-(s,q) \in (F(x) - y_0, G(x)) + (S \times Q)$. This means that $\emptyset \neq H_{y_0}(x) \cap [-(s,q) - (S \times Q)] = H_{y_0}(x) \cap (-C)$, a contradiction.

(iii) $\Rightarrow$ (iv). It is obvious.

(iv) $\Rightarrow$ (i). Assume the contrary. There exist $s \in S \setminus \{0\}$ and $x \in E$ such that $-s \in (F(x) - y_0)$. This means that there exists $z \in G(x) \cap (-Q)$ such that $-(s,0) = (-s,z) + (0,-z) \in [(F(x) - y_0, G(x)) + S \times Q] \cap [-(S \setminus \{0\}) \times \{0\}]$, a contradiction. ∎

Proposition 2.1 was proved in [4, 7, 8, 16] when Y and Z are finite dimensional spaces and F and G are single-valued functions.

We observe that if R_{y_0} and $-C$ are cone separated, i.e., there exists a pointed convex cone K such that $-C \setminus \{(0,0)\} \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$, then $(x_0, y_0) \in \operatorname{gr} F$ is a minimum solution of (P). We are interested in problems under what conditions R_{y_0} and $-C$ are cone separated. Such conditions, which guarantee the existence of a particular form of nonlinear separation functions (see [8]), will be called the image regular condition.

3. CONE SEPARATION AND OPTIMAL CONDITIONS

In this section, we provide a cone separation theorem and then apply it to study optimality conditions and duality for weakened convex and nonconvex set-valued functions in locally convex spaces. Namely, we consider the image regular condition

$$\overline{\operatorname{cone}}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}. \tag{3.1}$$

We observe that if $(x_0, y_0) \in \operatorname{gr} F$ is a minimum solution of (P), then $\operatorname{cone}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}$ by Proposition 2.1. On the other hand, if $x_0 \in E$ is a feasible point of problem (P), and $(x_0, y_0) \in \operatorname{gr} F$, then condition (3.1) implies that (x_0, y_0) is a Benson's proper minimum solution of (P). Indeed, assume the contrary. Then $\overline{\operatorname{cone}}(F(E) + S - y_0) \cap (-S) \neq \{0\}$, i.e., there exist $s \in S \setminus \{0\}$ and nets $\{\lambda_\alpha\}$ in R^1_+, $\{s_\alpha\}$ in S, $\{x_\alpha\}$ in A, and $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$, $G(x_\alpha) \cap (-Q) \neq \emptyset$ and $\lambda_\alpha(y_\alpha + s_\alpha - y_0) \to -s_0$. This means that there exists $z_\alpha \in G(x_\alpha) \cap (-Q)$ such that $\lambda_\alpha(y_\alpha + s_\alpha - y_0, z_\alpha - z_\alpha) \to (-s_0, 0)$. This contradicts (3.1).

THEOREM 3.1. *Assume that $(x_0, y_0) \in \operatorname{gr} F$ is a minimum solution of* (P), *and assume that either S has a weakly compact base and $\overline{\operatorname{cone}}(R_{y_0})$ is weakly closed or S has a compact base. Then R_{y_0} and $-C$ are cone separated, i.e., there exists a pointed convex cone K such that $-C \setminus \{(0,0)\} \subset$* int *$K$ and $R_{y_0} \cap K = \{(0,0)\}$, if and only if* (3.1) *is true.*

Proof. Assume that (3.1) is true. Since S has a compact (resp. weakly compact) base, we see that $S \times \{0\}$ has also a compact (resp. weakly compact) base. By Theorem 2.3 of [6], there exists a pointed convex cone P such that $-(S \times \{0\}) \setminus \{(0,0)\} \subset \operatorname{int} P$ and

$$\overline{\operatorname{cone}}(R_{y_0}) \cap P = \{(0,0)\}. \tag{3.2}$$

Set $K = (-C + P) \cup \{(0,0)\}$. Then K is a convex cone and $\operatorname{int} K = -C + \operatorname{int} P$. For every $s \in S \setminus \{0\}$, $q \in Q$, since $-(s,q) = -(s/2, q) - (s/2, 0) \in -(S \setminus \{0\} \times Q) + \operatorname{int} P = \operatorname{int} K$, we deduce that $-(C \setminus \{(0,0)\} = -C \subset \operatorname{int} K$.

We shall show that K is pointed. Since (x_0, y_0) is a minimum solution of (P), there exists $z_0 \in G(x_0) \cap (-Q)$. For every $(s,q) \in S \times Q$, we have $(s,q) = (0, z_0) + (s, q - z_0) \in (F(x_0) - y_0, G(x_0)) + S \times Q \subset R_{y_0}$. This means that $S \times Q \subset R_{y_0}$. It follows from $\overline{\operatorname{cone}}(R_{y_0}) \cap P = \{(0,0)\}$ that $(S \times Q) \cap P = \{(0,0)\}$ and so $C \cap P = \varnothing$. This, with the fact that S, Q, and P are pointed, implies that K is also pointed.

We need to show $R_{y_0} \cap K = \{(0,0)\}$. Indeed, assume the contrary. Then there exists $u \in R_{y_0} \cap K$, $u \neq 0$ and then $u = u_1 + u_2$ with $u_1 \in -C$ and $u_2 \in P$. This implies that $u_2 = u - u_1 \in (R_{y_0} - u_1) \cap P \subset \overline{\operatorname{cone}}(R_{y_0}) \cap P = \{(0,0)\}$. Hence, by Proposition 2.1, we have $u = u_1 \in (-C) \cap R_{y_0} = \varnothing$. It leads to a contradiction.

If there exists a pointed convex cone K such that $-C \setminus \{(0,0)\} \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$, then $R_{y_0} \cap \operatorname{int} K = \varnothing$ and then $\overline{\operatorname{cone}}(R_{y_0}) \cap \operatorname{int} K = \varnothing$. This, together with $-(S \setminus \{0\}) \times \{0\} \subset \operatorname{int} K$, implies that (3.1) is true. ∎

The scalar result corresponding to Theorem 3.1 can be found in Theorem 2.1 in [7]. Theorem 3.1 guarantees the existence of a class of weak separation functions as the following (see [8]).

THEOREM 3.2. *Assume that* $(x_0, y_0) \in \operatorname{gr} F$ *is a minimum solution of* (P), *and assume that either* S *has a weakly compact base and* $\overline{\operatorname{cone}}(R_{y_0})$ *is weakly closed or* S *has a compact base. Then there exists a continuous sublinear function* $\phi\colon Y \times Z \to R^1$ *satisfying*

(a) *for each* $z \in Z$, $\phi(\cdot, z)$ *is* S—*strictly increasing on* Y,

(b) *for each* $y \in Y$, $\phi(y, \cdot)$ *is* Q—*increasing on* Z *such that*

$$0 = \min_{x \in A} \phi\big(F(x) - y_0 + S, G(x) + Q\big)$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\phi(0, z_0) = 0$, *if and only if*

$$\overline{\operatorname{cone}}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}. \tag{3.1}$$

Proof. Assume that (3.1) is true. Then by Theorem 3.1, there exists a pointed convex cone K such that $-C \setminus \{(0,0)\} \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$. Take $u_0 \in \operatorname{int} K$ and define

$$\phi(u) := \inf\left\{\alpha \in R^1 \middle| u \in \left(\{-\alpha u_0\} + K\right)\right\}, \qquad u \in Y \times Z.$$

Theorem 3.1 in [6] implies that ϕ is a continuous sublinear functional on $Y \times Z$ which satisfies the properties

$$\operatorname{int} K = \{(y,z) | \phi(y,z) < 0\} \qquad \text{and} \qquad \overline{K} = \{(y,z) | \phi(y,z) \leq 0\}. \tag{3.3}$$

We shall show that (a) and (b) are satisfied.

(a) For every $y_2 - y_1 \in S \setminus \{0\}$, since $(y_1 - y_2, 0) \in -C \subset \operatorname{int} K$, (3.3) yields $\phi(y_1 - y_2, 0) < 0$. By the sublinearity of ϕ, for every $z \in Z$, we have $\phi(y_1, z) \leq \phi(y_1 - y_2, 0) + \phi(y_2, z) < \phi(y_2, z)$.

(b) For every $z_2 - z_1 \in Q$ since $(0, z_1 - z_2) \in -\overline{C} \subset \overline{K}$, (3.3) yields $\phi(0, z_1 - z_2) \leq 0$. By the sublinearity of ϕ, for every $y \in Y$, we have $\phi(y, z_1) \leq \phi(0, z_1 - z_2) + \phi(y, z_2) \leq \phi(y, z_2)$.

It follows from $R_{y_0} \cap K = \{(0,0)\}$ that

$$0 = \min_{x \in A} \phi\left(F(x) - y_0 + S, G(x) + Q\right).$$

For every $z_0 \in G(x_0) \cap (-Q)$, $(0, z_0) \in R_{y_0} \cap \overline{K}$ implies that $\phi(0, z_0) = 0$.

If there exists a continuous sublinear function ϕ: $Y \times Z \to R^1$ satisfying (a), (b) such that $0 = \min_{x \in A} \phi(F(x) - y_0 + S, G(x) + Q)$ and for every $z_0 \in G(x_0) \cap (-Q)$, $\phi(0, z_0) = 0$, and suppose that (3.1) is not true, then there exist $s \in S \setminus \{0\}$ such that $(-s, 0) \in \overline{\operatorname{cone}}(R_{y_0})$. Since $\phi(y,z) \geq 0$ for every $(y,z) \in R_{y_0}$, and ϕ is sublinear and continuous, we can deduce that $\phi(y,z) \geq 0$ for every $(y,z) \in \overline{\operatorname{cone}}(R_{y_0})$, and so $\phi(-s, 0) \geq 0$. On the other hand, the condition (a) implies that $\phi(-s, 0) < \phi(0,0) = 0$. It leads to a contradiction. ∎

A related result can be found in Theorem 6.4 of [6].

If $F \times G$ is nearly convexlike, then Theorem 3.1 gives the following Lagrangian multiplier theorem. We denote by $L^+(Z, Y)$ the set of all linear continuous operators Λ with $\Lambda(Q) \subset S$.

THEOREM 3.3. *Assume that* $(x_0, y_0) \in \operatorname{gr} F$ *is a minimum solution of* (P) *and* $\overline{R}_{y_0}$ *is convex, and assume that* S *has a weakly compact base. Then the following statements are equivalent*:

(i) $$\overline{\operatorname{cone}}\left(R_{y_0}\right) \cap \left(-S \times \{0\}\right) = \{(0,0)\};$$

(ii) *there exist* $y^* \in S^{+i}$, $z^* \in Q^+$ *such that*

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \qquad \forall (y,z) \in R_{y_0}, \tag{3.4}$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\langle z^*, z_0\rangle = 0$;

(iii) *there exists a continuous linear positive operator* $\Lambda \in L^+(Z,Y)$ *such that* (x_0, y_0) *is a Benson's proper minimum solution of the problem*

$$\min_{x\in A} (F(x) + \Lambda G(x)), \tag{$\overline{\mathrm{P}}$}$$

and for every $z_0 \in G(x_0) \cap (-Q)$, $\Lambda z_0 = 0$.

Proof. (i) → (ii). Since $\overline{R}_{y_0}$ is convex, we can deduce that $\overline{\mathrm{cone}}(R_{y_0}) = \overline{\mathrm{cone}}(\overline{R}_{y_0})$ is convex (see Proposition 4.21 of [1]). Thus it is weakly closed. By Theorem 3.1, there exists a pointed convex cone K such that $-C \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$, and then $\overline{R}_{y_0} \cap \operatorname{int} K = \varnothing$. By the standard separation theorem there exists $(y^*, z^*) \in -K^+$ such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \qquad \forall (y,z) \in R_{y_0}, \tag{3.4}$$

and

$$\langle y^*, y\rangle + \langle z^*, z\rangle < 0 \qquad \forall (y,z) \in \operatorname{int} K. \tag{3.5}$$

We shall show that $y^* \in S^{+i}, z^* \in Q^+$. It follows from (3.5) and $-((S\setminus\{0\}) \times Q) \subset \operatorname{int} K$ that

$$\langle y^*, y\rangle + \langle z^*, z\rangle < 0 \qquad \forall (y,z) \in -((S\setminus\{0\}) \times Q). \tag{3.6}$$

Since $0 \in Q$, we can deduce that $y^* \in S^{+i}$. Since $y \in S\setminus\{0\}$ can be made arbitrary close to 0 ($\in Y$), (3.6) implies that $z^* \in Q^+$. For every $z_0 \in G(x_0) \cap (-Q)$, (3.4) and $z^* \in Q^+$ imply that $\langle z^*, z_0\rangle = 0$.

(ii) → (iii). By (ii), there exist $y^* \in S^{+i}, z^* \in Q^+$ such that (3.4) hold and for every $z_0 \in G(x_0) \cap (-Q)$, $\langle z^*, z_0\rangle = 0$. Fix $e \in S\setminus\{0\}$ such that $\langle y^*, e\rangle = 1$. Define Λ: $Z \to Y$ by $\Lambda z = \langle z^*, z\rangle e$, for every $z \in Z$. Then $y^*\Lambda = z^*$, $\Lambda(Q) \subset S$, and $\Lambda z_0 = 0$. Substituting z^* by $y^*\Lambda$ in (3.4), we obtain that

$$\langle y^*, y + \Lambda z) \geq \langle y^*, y_0\rangle \qquad \forall (y,z) \in (F(x), G(x)). \tag{3.7}$$

Since $y_0 \in F(x_0) + \Lambda G(x_0)$ and $y^* \in S^{+i}$, we can conclude that (x_0, y_0) is a Benson's proper minimum solution of ($\overline{\mathrm{P}}$).

(iii) → (i). Assume that (3.1) is not true. Then there exist $s \in S\setminus\{0\}$, and nets $\{\lambda_\alpha\}$ in R^1_+, $\{s_\alpha\}$ in S, $\{q_\alpha\}$ in Q, $\{x_\alpha\}$ in A, $\{y_2\}$, and $\{z_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$, $z_\alpha \in G(x_\alpha)$, and $\lambda_\alpha(y_\alpha + s_\alpha - y_0, z_\alpha + q_\alpha) \to -(s,0)$. This means that $\lambda_\alpha(y_\alpha + \Lambda z_\alpha + s_\alpha + \Lambda q_\alpha - y_0) \to -s$ since Λ is continuous. The definition of Λ implies $\Lambda q_\alpha \in S$. Hence $(y_\alpha + \Lambda z_\alpha + s_\alpha + \Lambda q_\alpha$

$-y_0) \in (F + \Lambda G)(A) + S - y_0$, and so $-s \in \overline{\text{cone}}(F + \Lambda G)(A) + S - y_0)$. This contradicts the fact that (x_0, y_0) is a Benson's proper minimum solution of $(\overline{\text{P}})$. ∎

When X, Y, and Z are finite-dimensional spaces, and F and G are single-valued mappings, the conclusions that (iii) implies (i) and (i) implies (iii) in Theorem 3.3 improve Theorem 4.1 of [16] and Theorem 4.2 of [16], respectively.

When $\overline{R}_{y_0}$ and $\overline{F(E) + S}$ are convex and (x_0, y_0) is a Benson's proper minimum solution of (P), Theorem 3.3 was proved in [20] under the assumption of Slater's condition where the problem cannot involve equality constraints.

In the sequel of this section, we assume that X, Y, and Z are normed spaces and $A \subset X$ is a subset of X. For a given point $x \in \overline{A}$, the *contingent cone* $T_A(x)$ is defined by

$$T_A(x) = \left\{ v \in X \,\middle|\, \liminf_{h \to 0+0} h^{-1} d_A(x + hv) = 0 \right\},$$

here $d_A(x) = \inf_{y \in A} \|x - y\|$.

The *Clarke tangent cone* $C_A(x)$ is defined by

$$C_A(x) = \left\{ v \in X \,\middle|\, \limsup_{x' \to x,\, h \to 0+0} h^{-1} d_A(x' + hv) = 0 \right\}.$$

Assume that $F\colon X \to 2^Y$ is a set-valued function. Denote by $\operatorname{gr} F$, $\operatorname{dom} F$ the graph and domain of F.

$$\operatorname{gr} F = \{(x, y) \,|\, y \in F(x)\},$$
$$\operatorname{dom} F = \{x \,|\, F(x) \neq \varnothing\}.$$

For $(x, y) \in \operatorname{gr} F$, the *Clarke tangent derivative* $CF(x, y)\colon X \to 2^Y$ is defined by

$$\operatorname{gr} CF(x, y) = C_{\operatorname{gr} F}(x, y).$$

When F is single-valued, $CF(x, y) = CF(x, F(x))$.

A set-valued function F is called *locally Lipschitz* at $x_0 \in X$ if, for some constant l and some neighborhood U of x_0 such that for all $x_1, x_2 \in U$,

$$\rho(F(x_1), F(x_2)) \leq l \|x_1 - x_2\|,$$

where $\rho(\cdot, \cdot)$ denotes the Hausdorff distance (see [1]).

A set-valued function $F\colon X \to 2^Y$ is called *invex* at $(x_0, y_0) \in \operatorname{gr} F$ if $\hat{F}(X) - y_0 \subset \overline{C\hat{F}(x_0, y_0)(X)}$, where $\hat{F}(x) = F(x) + S$.

For set-valued functions $F: X \to 2^Y$, $G: X \to 2^Z$. Let $H(x) = (F(x), G(x))$, $\hat{H}(x) = H(x) + S \times Q$, $F \times G$ is called *invex* at (x_0, y_0, z_0) if H is invex at (x_0, y_0, z_0). For the definitions and some results of invex set-valued functions, we refer to [18, 19].

We denote by $F|_A$ the restriction of F to A, defined by

$$F|_A(x) = \begin{cases} F(x), & \text{if } x \in A; \\ \varnothing, & \text{otherwise}. \end{cases}$$

We assume that $A \subset \operatorname{dom} F = \operatorname{dom} G$, $A + \varnothing = \varnothing$, $A \times \varnothing = \varnothing$, and $\inf \varnothing = +\infty$.

THEOREM 3.4. *Assume that* $(x_0, y_0) \in \operatorname{gr} F$ *is a minimum solution of* (P), *and assume that either* S *has a weakly compact base and* $\overline{\operatorname{cone}}(R_{y_0})$ *is weakly closed or* S *has a compact base. Suppose that* F, G *are locally Lipschitz at* x_0, *and* $(F \times G)|_A$ *is invex at* (x_0, y_0, z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$. *Then the following statements are equivalent*:

(i) $\overline{\operatorname{cone}}(R_{y_0}) \cap (-S \times \{0\}) = \{(0,0)\}$;

(ii) *there exist* $y^* \in S^{+i}$, $z^* \in Q^+$ *such that*

$$\langle y^*, y \rangle + \langle z^*, z \rangle \geq 0 \qquad \forall (y, z) \in R_{y_0},$$

and $\langle z^*, z_0 \rangle = 0$;

(iii) *there exists a continuous linear positive operator* $\Lambda \in L^+(Z, Y)$ *such that* (x_0, y_0) *is a Benson's proper minimum solution of the problem*

$$\min_{x \in A} (F(X) + \Lambda G(x)), \tag{$\overline{\mathrm{P}}$}$$

and $\Lambda z_0 = 0$.

Proof. By Theorem 3.1, there exists a pointed convex cone K such that $-C \subset \operatorname{int} K$ and $R_{y_0} \cap K = \{(0,0)\}$, and then $\overline{R}_{y_0} \cap \operatorname{int} K = \varnothing$. By Theorem 5.3.1 of [1], we have that

$$\begin{aligned} &C(\hat{F} \times \hat{G})(x_0, y_0, z_0)(C_A(x_0)) \\ &\quad \subset T_{(\hat{F} \times \hat{G})(A)}(y_0, z_0) \\ &\quad \subset \overline{\operatorname{cone}}\big((\hat{F} \times \hat{G})(A) - (y_0, z_0)\big) \subset \overline{\operatorname{cone}}(R_{y_0}). \end{aligned}$$

Hence $\overline{C(\hat{F} \times \hat{G})(x_0, y_0, z_0)(C_A(x_0))} \cap \operatorname{int} K = \varnothing$. Since $C(\hat{F} \times \hat{G})(x_0, y_0, z_0)(C_A(x_0))$ is convex, by using the very similar arguments as in the proof of Theorem 3.3, we conclude that there exist $y^* \in S^{+i}$, $z^* \in Q^+$

such that

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0,$$

$$\forall (y, z) \in C(\hat{F} \times \hat{G})(x_0, y_0, z_0)(u),$$

$$u \in C_A(x_0) \cap \operatorname{dom} C(\hat{F} \times \hat{G})(x_0, y_0, z_0),$$

and $\langle z^*, z_0\rangle = 0$. Since $(\hat{F} \times \hat{G})|A$ is invex at (x_0, y_0, z_0), we deduce that

$$\begin{aligned}(\hat{F} \times \hat{G})(A) - (y_0, z_0) &\subset \overline{C(\hat{F} \times \hat{G})|_A(x_0, y_0, z_0)(X)}\\ &\subset \overline{C(\hat{F} \times \hat{G})(x_0, y_0, z_0)(C_A(x_0))}.\end{aligned}$$

Hence

$$\begin{aligned}\langle y^*, y\rangle + \langle z^*, z\rangle &\geq \langle y^*, y_0\rangle + \langle z^*, z_0\rangle\\ &= \langle y^*, y_0\rangle \qquad \forall (y, z) \in (\hat{F} \times \hat{G})(A)\end{aligned}$$

and so

$$\langle y^*, y\rangle + \langle z^*, z\rangle \geq 0 \qquad \forall (y, z) \in R_{y_0}.$$

Then, we repeat step by step the same argument as in the proof of Theorem 3.3. ∎

Sach and Craven [18] proved a similar result which was based on a weak minimum solution.

4. DUALITY

In this section, we shall consider the duality theorems. We first present a duality theorem of nonconvex problems.

We denote by Φ the set of all continuous sublinear functionals ϕ defined on $Y \times Z$ which satisfies properties

(a) for each $z \in Z$, $\phi(\cdot, z)$ is S—strictly increasing on Y,

(b) for each $y \in Y$, $\phi(y, \cdot)$ is Q—increasing on Z.

Define $\Psi\colon \Phi \to Y$ by

$$\Psi(\phi) = \left\{ y \in Y \,\middle|\, \exists x \in A \text{ s.t. } y \in F(x) \text{ and } 0 = \min_{\xi \in A} \phi(F(\xi) - y, G(\xi)) \right\}.$$

By a dual of (P), we mean the maximization problem

$$\max_{\phi \in \Phi} \Psi(\phi). \tag{D}$$

If $\phi \in \Phi$ is such that $\Psi(\phi) \neq \emptyset$, we call ϕ a *feasible point* of problem (D). If ϕ_0 is a feasible point of (P), $y_0 \in \Psi(\phi_0)$ and there is no feasible point ϕ of problem (D) such that

$$(y_0 - \Psi(\phi)) \cap (-S \setminus \{0\}) \neq \emptyset$$

we say that (ϕ_0, y_0) is a *maximum solution* of problem (D).

THEOREM 4.1. *Assume that either S has a weakly compact base and $\overline{\text{cone}}(R_{y_0})$ is weakly closed or S has a compact base. Assume that condition* (3.1) *holds.*

(i) *If x_0 is a feasible point of* (P) *and ϕ_0 is a feasible point of* (D), *then*

$$(F(x_0) - \Psi(\phi_0)) \cap (-S \setminus \{0\}) = \emptyset.$$

(ii) *If (x_0, y_0) is a minimum solution of* (P), *then there exists $\phi_0 \in \Phi$ such that (ϕ_0, y_0) is a maximum solution of* (D).

Proof. (i) If ϕ_0 is a feasible point of (D) then by the definition for every $y \in \Psi(\phi_0)$, there exists $x \in A$ such that $y \in F(x)$ and

$$0 = \min_{\xi \in A} \phi_0(F(\xi) - y, G(\xi)).$$

We shall show that $(F(x_0) - y) \cap (-S \setminus \{0\}) = \emptyset$. Assume the contrary. Then there exists $s_0 \in -S \setminus \{0\}$ such that $s_0 \in F(x_0) - y$. Since x_0 is a feasible point of (P), there is $z_0 \in G(x_0) \cap (-Q)$. Hence, $0 \leq \phi(s_0, z_0)$. Conditions (a) and (b) imply that $\phi(s_0, z_0) < \phi(0, z_0) \leq \phi(0, 0) = 0$. It leads to a contradiction.

(ii) If (x_0, y_0) is a minimum solution of (P), then by Theorem 3.2 there exists $\phi_0 \in \Phi$ such that

$$0 = \min_{\xi \in A} \phi_0(F(\xi) - y_0, G(\xi)).$$

This means that ϕ_0 is a feasible point of (D) and $y_0 \in \Psi(\phi_0)$. By (i), for every feasible point ϕ of (D), we have $(y_0 - \Psi(\phi)) \cap (-S \setminus \{0\}) = \emptyset$. Thus (ϕ_0, y_0) is a maximum solution of (D). ∎

Luc and Jahn [14] obtained a similar result based on Henig proper efficient solutions under a stronger condition.

Under the hypotheses of Theorem 3.3 and Theorem 3.4, we can obtain Lagrangian duality theorems. Namely, define the set-valued function $\overline{\Psi}$: $L^{+}(Z,Y) \to 2^{Y}$ by

$$\overline{\Psi}(\Lambda) = \left\{ y \middle| \exists \xi \in A \text{ s.t. } (\xi, y) \textit{ is a Benson's proper minimum solution of } (\overline{\mathrm{P}}) \right\}.$$

Consider the problem

$$\max\left\{ \overline{\Psi}(\Lambda), \Lambda \in L^{+}(Z,Y) \right\}. \qquad (\overline{\mathrm{D}})$$

Using similar arguments as above, we can prove the following duality results:

THEOREM 4.2. *Assume that* $\overline{R}_{y_0}$ *is convex, that* S *has a weakly compact base, and that condition* (3.1) *holds.*

(iii) *If* Λ_0 *is a feasible point of problem* $(\overline{\mathrm{D}})$ *and* x_0 *is a feasible point of problem* (P), *then*

$$\left(F(x_0) - \overline{\Psi}(\Lambda_0) \right) \cap (-S \setminus \{0\}) = \varnothing;$$

(iv) *If* (x_0, y_0) *is a minimum solution of* (P), *then there exists* $\Lambda_0 \in L^{+}(Z,Y)$ *such that* Λ_0 *is a feasible point of* $(\overline{\mathrm{D}})$ *and* (Λ_0, y_0) *is a maximum solution of* $(\overline{\mathrm{D}})$.

When $\overline{R}_{y_0}$ and $\overline{F(E) + S}$ are convex and (x_0, y_0) is a Benson's proper minimum solution of (P), Theorem 4.2 was proved in [20] under the assumption of Slater's condition.

THEOREM 4.3. *Let* X, Y, *and* Z *be normed spaces. Assume that either* S *has a weakly compact base and* $\overline{\mathrm{cone}}(R_{y_0})$ *is weakly closed or* S *has a compact base. Suppose that* F, G *are locally Lipschitz at* x_0, *and* $(F \times G)|_A$ *is invex at* (x_0, y_0, z_0) *for some* $z_0 \in G(x_0) \cap (-Q)$, *and suppose that condition* (3.1) *holds.*

(iii) *If* Λ_0 *is a feasible point of problem* $(\overline{\mathrm{D}})$ *and* x_0 *is a feasible point of problem* (P), *then*

$$\left(F(x_0) - \overline{\Psi}(\Lambda_0) \right) \cap (-S \setminus \{0\}) = \varnothing;$$

(iv) *If* (x_0, y_0) *is a minimum solution of* (P), *then there exists* $\Lambda_0 \in L^+(Z, Y)$ *such that* Λ_0 *is a feasible point of* $(\overline{\mathrm{D}})$ *and* (Λ_0, y_0) *is a maximum solution of* $(\overline{\mathrm{D}})$.

Sach and Craven [18] proved a similar result which was based on a weak minimum solution.

ACKNOWLEDGMENTS

The author thanks Professor S. Rolewicz for his helpful discussion and Professor Koichi Mizukami for some corrections.

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